MATH 1010A Notes
Week 6 (Day 1)

Introduction
Topics to cover

- Equation of Tangent Line (pending, cos not appeared in A1-A3)
- Implicit Differentiation
- MVT

Implicit Diff

Having discussed "almost" all rules about derivatives, we mention one more (very very important) rule:

Given an equation of the form

$$f(x, y) = 0$$

Where on the LHS we have a "function of two variables" and on the RHS a "constant".

Can we differentiate y w.r.t. ? If "yes", what is the reason behind?

Answer:

Yes. Reason = "Implicit Function Theorem". In school, we learned that the <u>equation</u> $x^2 + y^2 = a^2,\ a > 0$ defines a circle.

We can make y the "subject" to get $y = \pm\sqrt{x^2 - a^2}$ hence y is a function of x.

In the same way, $x = \text{function(s) of } y$.

Notation

Let's denote "y is a function of x" by $y = y(x)$. Then we obtain after some work

$y' = -\frac{y}{x}$

The Implicit Function Theorem says that "if $f(x, y) = c$, then $y = y(x)$ or $x = x(y)$" (if some assumptions are made on the function f).

Using this, we can find y', x' in, for example, $(x^2 + y^2)^2 - x^2 + y^2 = 0$

Application

One good application of the above is

Example

Find $\frac{d(\arcsin x)}{dx}$.

Solution:

Ask the question: "What is arcsin(x)?" The answer (without worrying about 1-1, range etc.) is:

$$x = \sin(y), \text{ then } \quad y = \arcsin x$$

The equation on the left-hand side can be rewritten in the form $f(x, y) = c$. To see this, do the following:

$$\sin(y) - x = 0$$

Where the LHS is of the form $f(x, y)$.

Now use implicit D to get

$$\frac{d \sin y}{dx} - \frac{dx}{dx} = 0$$

Hence giving

$$\frac{d \sin y}{dy} \frac{dy}{dx} - 1 = 0$$

$$\frac{d \sin y}{dy} \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{\cos y}$$

The remaining problem is to "rewrite" the RHS in terms of x. This is done by using

$$x = \sin(y), \text{i.e. } 1 - x^2 = \cos^2(y) \text{ i.e.} \cos(y) = \pm\sqrt{1 - x^2}$$

Putting this back into the formula for $\frac{dy}{dx} = \frac{1}{\cos y}$ gives

$$\frac{dy}{dx} = \pm\frac{1}{\sqrt{1-x^2}}$$

Mean Value Theorems

One great applications of finding derivatives is Taylor's Theorem (our next goal), mentioned a long time ago. To get that one, we need the following

Rolle's Theorem

Let $f: [a,b] \to R$ be (i) continuous at each point in $[a,b]$, (ii) $f(x)$ is differentiable at each interior point, (iii) $f(a) = f(b)$.

Then there is at least one point in (a,b) so that $f'(c) = 0$.

Related to this is the LMVT

Lagrange's MVT

If in the above theorem, we remove condition (iii), then we get

$$\frac{f(b)-f(a)}{b-a} = f'(d)$$

Application

A nice application is this:

Claim: If $f(x)$ satisfies $f'(x) > 0$ at each point in its domain, say (a,b), then $f(s) < f(t)$ whenever $s < t$.

Solution: Use LMVT for the domain $[s,t]$. ($[s,t]$ is a subset of $[a,b]$, so it is also a domain).

Then we have $\frac{f(t)-f(s)}{t-s} = f'(p)$ for some $p \in (s,t)$. But now $f'(p) > 0$ so the fraction on the LHS must be > 0. This gives

$$\frac{f(t)-f(s)}{t-s} > 0$$

Remembering that we have $t > s$, so the denominator is > 0. This forces the numerator to be also > 0.

Hence we have shown: "Whenever $t > s$, then $f(t) > f(s)$."